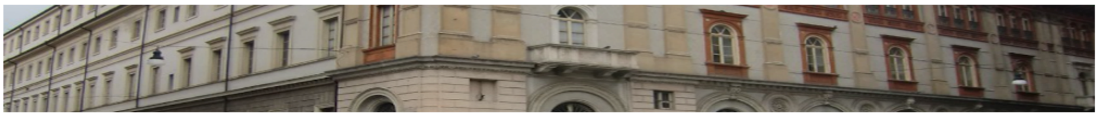
Theory and Computation in Algebra and Algebraic Geometry



With a dedication to Paolo Valabrega on the occasion of his 70(+2)th Birthday

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Combinatorics of involutive divisions

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Denote by $\mathcal{P} := \mathbf{k}[x_1, ..., x_n]$ the graded ring of polynomials in n variables with coefficients in the field $\mathbf{k}$, $char(\mathbf{k}) = 0$ and by $\mathcal{T} := \{x^\gamma := x_1^{\gamma_1} \cdots x_n^{\gamma_n} \mid \gamma = (\gamma_1, ..., \gamma_n) \in \mathbb{N}^n\}$ the *semigroup of terms* generated by the set $\{x_1, ..., x_n\}$.

Given a monomial/semigroup ideal $J \subset \mathcal{T}$ and its minimal set of generators $\mathsf{G}(J)$ (also called its *monomial basis*), Janet introduced in [19] both the notion of *multiplicative variables* and the connected decomposition of J into disjoint *cones*. Then, he gave a procedure (*completion*) to produce such a decomposition.

In the same paper, in order to describe Riquier's [24] formulation of the description for the general solutions of a PDE problem, Janet gave a similar decomposition in terms of disjoint cones, generated by multiplicative variables, also for the related normal set/order ideal/*escalier* $\mathbf{N}(J) := \mathcal{T} \setminus J$.

Later in [20, 21, 22], he gave a completely different decomposition (and the related algorithm for computing it) which labelled as *involutive* and which is behind both Gerdt-Blinkov [9, 10, 11] procedure for computing Gröbner bases and Seiler's [27] theory of involutiveness.

The aim of Janet in these three papers was twofold:

1. to reinterpret, in terms of multiplicative variables and cone decomposition, the solution of PDE problems given by Cartan [1, 2, 3], whence the name *inolutiveness*;

2. to re-evaluate within his theory the notion of *generic initial ideal* introduced by Delassus [4, 5, 6] and the correction of his mistake by Robinson [25, 26] and Gunther [14, 15], who point out that the notion requires J to be Borel-fixed (a modern but identical reformulation was proposed by Galligo [8], which merged the considerations of Hironaka [18] and Grauert [12]; see also [13] and [7]); Janet remarked that all Borel-fixed ideals are involutive, but the converse is false.

More precisely, in his survey [21] Janet presents, as *nouvelle formes cannoniques*, the results of Delassus, Robinson and Gunther and compares them with the one which can be deduced from an involutive basis; and in [22, p.62], assuming to have a given a homogeneous ideal $\mathcal{I} \subset \mathcal{P}$ within a generic frame of coordinates, he reformulates Riquier's completion proposing essentially a Macaulay-like construction, iteratively computing the vector-spaces $\mathcal{I}_d := \{f \in \mathcal{I} : \deg(f) = d\}$ until Cartan test grants that Castelnuovo-Mumford [23, pg.99] regularity D has been reached. This would allow him to consider the monomial ideal $\mathbf{T}(\mathcal{I})$ of the leading terms (in the sense of Gröbner basis theory) w.r.t. a deg-lex term-ordering and obtain the related involutive reduction required by Riquier's procedure.

The results on involutiveness presented in both papers, however, simply restate the results of [20] which reinterprets Cartan's result in terms of multiplicative variables; more precisely Janet assumes to have a set of forms of degree D which satisfies Cartan test and directly considers both the monomial ideal

$$\mathsf{T} := \mathbf{T}(\mathcal{I}) \subset \mathcal{T}_{\geq D} =: \{t \in \mathcal{T}, deg(t) \geq D\}$$

and the partial *escalier*

$$\mathsf{N} := \mathcal{T}_{\geq D} \setminus \mathsf{T} = \{\tau \in \mathbf{N}(\mathcal{I}) : deg(\tau) \geq D\}$$

decomposing both of them in terms of disjoint cones, generated by multiplicative variables. Actually, he simply explicitly demotes the rôle of the ideal in this construction[1] considering the whole set $\mathcal{T}_{\geq D}$ and decomposing it in terms of disjoint cones generated by multiplicative variables, the related set of vertices being the set of all monomials $\mathcal{T}_D = \{\tau \in \mathcal{T} : \deg(\tau) = D\}$.

The aim of this paper is to discuss involutiveness following the approach proposed by Janet in [20]; in particular we postpone the discussion of ideal membership and related test only after having performed a deep reconsideration of the combinatorial properties of involutive divisions [9, 10, 11], when defined on the set $\mathcal{T}_D$.

To do so, we of course apply the theory of involutive divisions, set up by Gerdt–Binklov [9, 10, 11], but we are forced to slightly adapt it, talking about *relative* involutive divisions, and requiring that the union of all the cones produces the ideal $\mathcal{T}_{\geq D}$ and that the cones are disjoint; in fact our setting considers the *single* finite set $\mathcal{T}_D$ and thus does not require (as, of course, they need) comparing different divisions.

Moreover, the aim of their theory is to produce a setting for describing and building a Riquier-Janet procedure for computing Gröbner-like bases for ideals; thus they cannot assume neither that the division is involutive, *i.e.* that the union of all the cones defined on a set U produces the ideal generated by U (this being in their setting the aim of the procedure) nor uniqueness of involutive divisors, *i.e.* that that all cones are disjoint (the failure of this condition triggering the completion procedure). These two conditions are instead essential to grant that (the implicit procedure is completed and that) a unique decomposition is available both for the given ideal (granting unique reduction) and its associated *escalier* (granting standard Hironaka-like description of canonical forms).

We discuss the combinatorial structure of relative involutive divisions; we begin with the combinatorial formula given by Janet [19, 20, 21] and Gunther [16, p.184] evaluating, for each $i, 1 \leq i \leq n$, the number σ_i of the cones having i multiplicative variables and which is, essentially, an adaptation of Vandermonde's convolution [32, pg.492]; next we prove a set of Lemmata, which allows us to sketch an approach for imposing a relative involutive division structure on $\mathcal{T}_{\geq D}$ and which will be generalized to a procedure to list all the possible relative involutive divisions up to symmetries

We further characterize the relative involutive divisions which are Pommaret divisions up to a relabelling of the variables.

Thus, given a complete description of the combinatorial structure of a relative involutive divisions, we turn our attention to the problem of membership. Let us begin with the trivial remarks that if a term $u \in \mathcal{T}_D$ is contained (or is a generator) of the monomial ideal T,

- the whole cone whose vertex is u is contained in the ideal and that
- for each non-multiplicative variable x, there is necessarily a term $v \in \mathcal{T}_D$, $v \neq u$ s.t. xu belongs to the cone whose vertex is v and that such vertex (and cone) necessarily belongs to T;
- conversely, if v belongs to N, not only its related cone belongs to N, but the same holds to u and its related cone.

Moreover if T is not trivial then both

- the single monomial m which has no non-multiplicative variables, and its cone necessarily belong to T while
- for at least a value $i, 1 \leq i \leq n$, $x_i^D \in \mathsf{N}$.

On the basis of these remarks, we can define on $\mathcal{T}_D$ a rooted directed graph whose root is m and where an arrow $u \to v$ is given when, for a non multiplicative variable x_i for u, $x_i u$ belongs to the cone whose vertex is v. Of course such graph is redundant and our aim is to give a (more compact, non necessarily minimal) directed graph which has the following properties:

[1] *La proposition est vraie en particuier pour le système involutif constitué par* tout *les monomes d'ordre [D].* [20, p.46]

- if a vertex h is included in T and we walk against the flow toward the "peak" m, we reach all the terms in $\mathcal{T}_D$ which necessarily belong to T too; and
- if a vertex n is included in N and we follow the flow toward the "mouths" we reach all the terms in $\mathcal{T}_D$ which necessarily belong to N too.

We begin our investigation giving conditions (based on the computation of $\mathrm{lcm}(s,t)$, $s \in U$, which, for each $t \in \mathsf{T}$ (resp. N) allows to deduce further elements $X(t,s)$ (actually the vertex of the cone containg $\mathrm{lcm}(s,t)$) which are necessary members of T (resp. N).

Next we specialize our investigation to Pommaret divisions for which we prove that it is sufficient to adapt Ufnarovsky graph [28, 29, 30, 31] to the commutative case in order to obtain a graph which has exactly the shape and properties described above.

Unfortunately, in general, a graph with the properties described above cannot exist; in fact we show an example in n variables and degree $d = n-1$, in which the d monomials with $n-1$ multipicative variables are connected together, via their single non-multiplicative variables, in a loop which walks around the "peak" m; moreover these n monomials having either n or $n-1$ multiplicative variables form, if we can say so, a "canton", in the sense that if each of them belongs to either T or N the same happens for all of them.

Thus, in general, it is impossible to produce a graph as the Ufnarovsky-like existing for Pommaret division and which has the required structure, simply by multiplying each monomial t by its non-mutiplicable variables x and the graph is obtained recording the cone in which xt belongs.

The only way we are seeing for producing a graph with the required properties is to build the rendundant graph which can be obtained by testing the condition on $\mathrm{lcm}(s,t)$ and extract a minimal subgraph, an approach which in general is NP-complete.

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